On the sandpile group of a graph

Robert Cori,
Labri, Université Bordeaux 1 and Lix, Ecole Polytechnique
Dominique Rossin
Lix, Ecole Polytechnique

Abstract

We show how to express the sandpile model, introduced in theoretical physics, using the vocabulary of combinatorial theory. The group of recurrent configurations in the sandpile model, introduced by D. Dhar ([6]), may be considered as a finite abelian group associated with any graph G; we call it the sandpile group of G. The structure of the sandpile group is determined for some families of graphs.

Introduction

Self-organized criticality is a concept widely considered in various domains since Bak, Tang and Wiesenfeld introduced it ten years ago [2]. An overview of the applications of this concept is given in a recent book [1]. One of the paradigms in this framework is the Abelian Sandpile Model, introduced by D. Dhar [6] who pointed out some important mathematical properties of this model.

The abelian sandpile model could be described informally as a cellular automaton on a graph as follows :

The cells of such automata are the vertices of a rooted graph and each cell contains a certain number of grains of sand. The transitions of the automaton are given by the following rule called the *toppling rule*, which is applied to any cell except the root: *a cell* x_i *containing at least as many grains as its degree* d_i *transfers a grain of sand to each of its neighbors* x_j.

After a toppling of the vertex x_i the number of grains in this cell hence decreases by its degree while the number of those of its neighbors increases by 1. The root does not topple and could be considered as collecting all the grains leaving the system. If the graph is connected it is easy to see that from any initial configuration the system reaches a *stable* configuration in which the number of grains in each cell is less than its degree.

D. Dhar remarked also that some configurations, so-called *recurrent configurations,* play an important role and possess some interseting mathematical properties: they form a finite abelian group whose order is equal to the number of spanning trees of the graph.

The sandpile automaton was also studied by many authors, [4], [9], [10], [14]; some called it the *chip-firing game* on a graph.

Our aim in this paper is to give a simple introduction to the Abelian Sandpile Model from a combinatorial perspective. We also give some results on the structure of the group of recurrent configurations for planar graphs, n-wheels and complete graphs. In the last case we show a close relationship between recurrent configurations and the Parking Functions considered by many combinatorialists [8], [15], [16].

The paper is organized as follows :

In Section 1, we consider for any graph G a subgroup of $\mathbb{Z}^n$, where n is the number of vertices of G; the elements of $\mathbb{Z}^n$ may be considered as configurations in which the number of grains of sand in each cell may be positive or negative. The subgroup is generated by n elements expressing the toppling rules. The fact that the quotient of $\mathbb{Z}^n$ by this subgroup is an abelian group, whose order is number of spanning trees of G, is a reformulation of the classical *Matrix Tree Theorem.*

In Section 2 we consider configurations with a non-negative number of grains in each cell, and recall the definition of the recurrent configurations introduced by D. Dhar. We show that there is exactly one recurrent configuration in each class of the quotient group. This result is contained in [6], [7] but presented here in a different form.

In the other three sections we give some results on the structure of this quotient group. We show that the group of a planar graph and that of its duals are isomorphic (Section 3). In Section 4, we show that the group of the n-wheel is a product of two cyclic groups whose orders are related to the Fibonacci numbers. In the last Section we compute the groups of the complete graphs K_n and $K_{n,n}$, showing that they are the direct product of $n-2$ and $2n-3$ cyclic groups respectively; and we also give a bijection between recurrent configurations of K_n and parking functions.

1 Preliminaries

Let $G=(X,E)$ be a *multi-graph*; $X=\{x_1,\dots,x_n\}$ is the vertex set, and E is a symmetric $n\times n$ matrix such that $e_{i,j}$ is the number of edges with endpoints x_i, x_j. We will assume that for any i, $e_{i,i}=0$ so that the multi-graph has no loops. In most of the examples considered, $e_{i,j}$ is either 0 or 1, and G is simply a graph. The *degree* of the vertex x_i in G, denoted by d_i, is equal to $\sum_{j=1}^{n} e_{i,j}$. A multi-graph is *rooted* if one of its vertices is distinguished as the root; if not otherwise mentioned we will assume that the vertices are numbered in such a way that x_n is the root. Throughout this paper we assume that all the graphs considered are *connected graphs.*

A *configuration* u on G is an assignment of integers (positive or negative) to the vertices of the graph. Such a configuration will be denoted :

$$u=(u_1,\dots,u_n)$$

where u_i is the integer assigned to the vertex x_i. The set of configurations forms a free Abelian group with respect to the addition of two elements; it is isomorphic to $\mathbb{Z}^n$ and generated by the configurations $x_i = (0,\dots,0,1,0,\dots,0)$, (where 1 is in position i) for $i=1,\dots,n$.

In this paper we are mainly concerned with the subgroup $\Delta(G,x_n)$ of $\mathbb{Z}^n$ generated by x_n and the following elements Δ_i, $i=1,\dots,n$ given by :

$$\Delta_i = d_i x_i - \sum_{j=1}^{n} e_{i,j} x_j$$

We will write :

$$\Delta(G,x_n)=\langle x_n,\Delta_1,\dots,\Delta_n\rangle$$

Note that since $\sum_{i=1}^{n}\Delta_i=0$, we may discard one of the Δ_i in the above definition.

A classical result in algebraic graph theory, often called the *matrix tree theorem*, (see for instance [3] Theorem 6.3) states that the number of spanning trees of G is given by any principal minor of the matrix whose rows are the

Δ_i. Another way to express this theorem is to say that the quotient group $\mathbb{Z}^n/\Delta(G, x_n)$ is finite and its order is equal to the number of spanning trees of G. We call this quotient group the *sandpile group* of G and we will denote it by $SP(G, x_n)$. The name sandpile follows the physical model which is considered in the Introduction.

We first prove the following :

Proposition 1.1. *For any multi-graph G the sandpile group $SP(G, x_n)$ is independent of the root x_n.*

Proof We will prove that $SP(G, x_n)$ is isomorphic to $SP(G, x_1)$. Remark that the subgroup

$$\Delta(G, x_n) = \langle x_n, \Delta_1, \ldots, \Delta_n \rangle$$

is also generated by x_n and the following elements Δ'_i $(i = 2, \ldots, n)$:

$$\Delta'_i = \Delta_i + e_{i,1} x_n$$

Since $d_i = \sum_{j=1}^n e_{i,j}$ we may write Δ'_i as :

$$\Delta'_i = d_i(x_i - x_1) - \sum_{j=1}^{n} e_{i,j}(x_j - x_1) \ + e_{i,1} x_n$$

The following n elements $y_1, \ldots, y_n$ form a set of generators of $\mathbb{Z}^n$, since it is easy to express the x_i as linear combinations of the y_i :

$$y_1 = -x_n, \ y_2 = x_2 - x_1 \ , y_3 = x_3 - x_1, \ \ldots, \ y_n = x_n - x_1$$

Expressed in terms of basis formed by the y_i the Δ'_i become:

$$\Delta'_i = d_i y_i - \sum_{j=2}^{n} e_{i,j} y_j - e_{i,1} y_1$$

But this is exactly the definition of Δ_i in which x_i is replaced by y_i. Hence :

$$\Delta(G, x_n) = \langle y_1, \Delta'_2, \ldots, \Delta'_n \rangle$$

is isomorphic to $\Delta(G, x_1)$, giving the result. □

As a consequence of Proposition 1.1 we may denote by $SP(G)$ the sandpile group of the graph G, and omit the root in this notation.

We consider now an example of a graph for which we compute the sandpile group.

Example : The graph W_5 below, called the *5-wheel*, consists of five vertices $x_1, \ldots, x_5$ forming a cycle and all joined to another vertex x_6 which is the root.

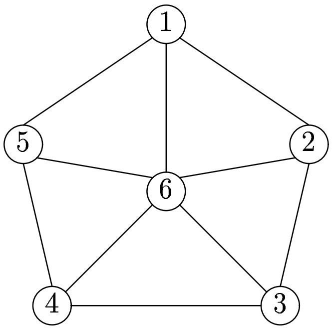

Figure 1: The 5-wheel W_5

The group $\Delta(W_5, x_6)$ is generated by the elements :

$$x_6,\ \ 3x_1 - x_5 - x_2,\ \ 3x_2 - x_1 - x_3,\ \ 3x_3 - x_4 - x_2,\ \ 3x_4 - x_3 - x_5,\ \ 3x_5 - x_4 - x_1$$

Let $\overline{x}_i$ be the image of x_i in the group $SP(G)$, we have :

$$\overline{x}_3 = 3\overline{x}_2 - \overline{x}_1,\ \ \overline{x}_4 = 3\overline{x}_3 - \overline{x}_2,\ \ \overline{x}_5 = 3\overline{x}_4 - \overline{x}_3,\ \ \overline{x}_1 = 3\overline{x}_5 - \overline{x}_4,\ \ \overline{x}_2 = 3\overline{x}_1 - \overline{x}_5,$$

Eliminating $\overline{x}_3, \overline{x}_4, \overline{x}_5$ gives the two following relations :

$$\left\{ \begin{array}{lcl} 55\overline{x}_2 & = & 22\overline{x}_1 \\ 143\overline{x}_2 & = & 55\overline{x}_1 \end{array} \right.$$

From these we get :

$$11\overline{x}_2 = 0 \quad 11(2\overline{x}_2 - \overline{x}_1) = 0$$

Hence the group $SP(W_5)$ is the direct product of two cyclic groups of order 11 generated by $\overline{x}_2$ and $2\overline{x}_2 - \overline{x}_1$.

In Sections 4 and 5 we will compute the structure of sandpile groups for some families of graphs using the classical algorithm for the computation of a group quotient of $\mathbb{Z}^n$ by a set of relations $\rho_1, \ldots, \rho_p$ which is described in many text books [5], [13]. An efficient version of this algorithm is given in [12]. It mainly consists of performing one of the following transformations :

- Replace a generator u_i with $u_i + au_j$, where a is an integer and u_j is another generator $(i \neq j)$.
- Replace a relation ρ_i with $\rho_i + a\rho_j$, where a is an integer and ρ_j is another relation $(i \neq j)$.

The algorithm stops when all the relations are of the form $\rho_i^* = a_i u_j^*$, for $i = 1, \dots, q$; this shows that the group is isomorphic to

$$\mathbb{Z}/a_1\mathbb{Z} \times \mathbb{Z}/a_2\mathbb{Z} \times \cdots \times \mathbb{Z}/a_q\mathbb{Z}$$

The matrix version of this algorithm is usually called the computation of the *Smith Normal Form.*

2 Recurrent configurations

In the model of the sand-pile automaton the number of grains in each cell is of course assumed to be non-negative; this leads to the consideration of *positive* configurations u where $u_i \geq 0$ for any $i = 1, \dots, n-1$. The toppling rule for cell i is equivalent to the subtraction of Δ_i from the configuration u. This can be formalized as follows :

For two positive configurations u and v, we denote

$$u \longrightarrow v$$

if there exists an $i \leq n-1$ such that $v = u - \Delta_i$. The configuration v is said to be obtained from u by *toppling* vertex x_i; we assume that the root does not topple. We denote by $\xrightarrow{*}$ the transitive closure of $\longrightarrow$, so that if $u \xrightarrow{*} v$ we have $u - v \in \Delta(G, x_n)$, hence u and v have the same image in the group $SP(G)$. A positive configuration is said to be *stable* if no vertex can topple, that is if $u_i < d_i$ for all vertices $i = 1, \dots, n-1$.

Proposition 2.1. *For any positive configuration u there exists a stable configuration v such that $u \xrightarrow{*} v$. Moreover this configuration is unique.*

Proof To prove existence, we consider the decomposition of the vertices of the graph G induced by the distance to the root x_n : we denote by X_k $(k = 0, \dots, d)$ the set of vertices whose distance to x_n is k, hence $X_0 = \{x_n\}$, X_1 is the set of neighbors of x_n, and so on.

To any configuration u we associate the $(d+1)$-tuple $\mu(u)$ of integers $\mu = \mu_0, \mu_1, \ldots, \mu_d$ given by :

$$\mu_k = \sum_{i \in X_k} u_i$$

We consider the following lexicographic order $\prec$ on these d-tuples : $\mu \prec \nu$ if there exists an integer k, $0 \leq k \leq d$ such that:

$$\mu_j = \nu_j \quad \text{for } j = 1, \ldots, k-1 \quad \text{and} \quad \mu_k < \nu_k$$

It is clear that $u \xrightarrow{*} v$ implies $\mu(u) \prec \mu(v)$, moreover the sums $|u| = \sum_{i=0}^{d} \mu(u)_i$ and $|v| = \sum_{i=0}^{d} \mu(v)_i$ are equal. Since there is no infinite ascending chain for $\prec$ in which all elements have the same sum of coordinates, we have the existence part of the Proposition.

To prove uniqueness, it suffices to show that the toppling sequence " cell i then cell j" is equivalent to that of "cell j then cell i", but this is a consequence of :

$$(u - \Delta_i) - \Delta_j = (u - \Delta_j) - \Delta_i$$

□

A configuration is *recurrent* in an evolving system if it could be observed after a long period of the evolution of the system. In the case of the Abelian sandpile model, the system is considered to evolve by adding a grain of sand in any cell at random and then applying toppling rules until a stable configuration is reached. This translates into the following notion which is central :

Definition 1. *A configuration u is recurrent if it is stable and there exists a positive configuration $v \neq 0$ such that $u + v \xrightarrow{*} u$.*

Our aim is now to prove that for any configuration u there exists a unique recurrent configuration v such that $u - v \in \langle \Delta_1, \Delta_2, \ldots, \Delta_n \rangle$.

We will use the configuration δ such that $\delta_i = d_i$ for all vertices; remark that for any stable configuration u, $\delta - u$ is a positive configuration.

The following simple remarks will be useful :

Remarks:

1. If u, v, u', v' are positive configurations satisfying $u \xrightarrow{*} u'$ and $v \xrightarrow{*} v'$ then $u + v \xrightarrow{*} u' + v'$.

2. For any positive configuration u there exists an integer k and a configuration v (not necessarily stable) such that :

$$ku \xrightarrow{*} v \quad \text{and} \quad v_i > 0 \text{ for } 1 \leq i \leq n-1$$

3. A configuration u is recurrent if and only if there exists a positive configuration u' such that

$$\delta + u' \xrightarrow{*} u$$

We will also need some new notation. For two positive configurations u and v we denote by $u \oplus v$ the unique stable configuration such that :

$$u + v \xrightarrow{*} u \oplus v$$

For two (not necessarily positive) configurations we write

$$u \dashrightarrow v$$

if there exists a vertex x_i such that $v = u - \Delta_i$, and $\overset{*}{\dashrightarrow}$ denotes the transitive closure of $\dashrightarrow$.

The following Lemma is our first step in order to prove the main result of this section :

Lemma 2.2. *Let u and v be two configurations such that*

$$u - v \in \langle \Delta_1, \Delta_2, \ldots, \Delta_n \rangle$$

then there exists a configuration w satisfying :

$$w \overset{*}{\dashrightarrow} u \quad \text{and} \quad w \overset{*}{\dashrightarrow} v$$

Proof With the hypothesis of the Lemma we may write

$$u = v + \sum_{i=1}^{n} \alpha_i \Delta_i$$

where α_i are integers. We may decompose the set $\{1, \ldots, n\}$ into two disjoint subsets I and J such that $I = \{i | \alpha_i \geq 0\}$ and $J = \{i | \alpha_i < 0\}$; then we have :

$$u - \sum_{i \in J} \alpha_i \Delta_i = v + \sum_{i \in I} \alpha_i \Delta_i$$

It is then clear that the configuration $w = u - \sum_{i \in J} \alpha_i \Delta_i$, satisfies the conditions $w \overset{*}{\dashrightarrow} u$ and $w \overset{*}{\dashrightarrow} v$. □

Let ε be the configuration

$$\varepsilon = \delta + (\delta \; - \; \delta \oplus \delta)$$

Then we have

Lemma 2.3. *The configurations ε and δ satisfy :*

$$\delta + \varepsilon \xrightarrow{*} \delta$$

Proof We have by definition of ε :

$$\delta + \varepsilon = \delta + \delta + (\delta - \delta \oplus \delta)$$

Since $(\delta - \delta \oplus \delta)$ is a positive configuration we have by Remark 1 :

$$\delta + \varepsilon \xrightarrow{*} \delta \oplus \delta + (\delta - \delta \oplus \delta)$$

but this last configuration is obviously equal to δ.

□

Lemma 2.4. *A configuration u is recurrent if and only if :*

$$u + \varepsilon \xrightarrow{*} u$$

Proof If $u + \varepsilon \xrightarrow{*} u$, u is recurrent since ε is positive. Conversely, if u is recurrent there exists (by Remark 3) a positive configuration v such that $v + \delta \xrightarrow{*} u$, hence by Remark 1 :

$$v + \delta + \varepsilon \xrightarrow{*} u + \varepsilon$$

but by Lemma 2.3 we have

$$v + \delta + \varepsilon \xrightarrow{*} v + \delta \xrightarrow{*} u$$

Since u is stable, using Proposition 2.1 we have $u + \varepsilon \xrightarrow{*} u$ and hence the result □

We are now able to prove the main Theorem of this section :

Theorem 1. *For any configuration u there exists a unique recurrent configuration v such that :*

$$u - v \in \quad \langle \Delta_1, \Delta_2, \dots, \Delta_n \rangle$$

Proof In order to simplify the notation let us denote $\langle \Delta_1, \Delta_2, \dots, \Delta_n \rangle$ simply by Δ. We first prove existence. We have trivially $\varepsilon \in \Delta$ since $\delta \oplus \delta = \delta + \delta + t$ where $t \in \Delta(G, x_n)$, hence $\varepsilon = -t$ is also in $\Delta(G, x_n)$. Now ε has all its components strictly positive, hence for a certain $k > 0$ the components of $k\varepsilon$ are greater than all the absolute values of those of $u - \delta$, and $u + k\varepsilon$ may be written as $\delta + u'$, where u' is a positive configuration. Then there exists a stable configuration v such that $u + k\varepsilon \xrightarrow{*} v$. The configuration v is recurrent by Remark 3, and it clearly satisfies the condition of the Theorem.

To prove uniqueness we show that if u and v are two recurrent configurations such that $u - v \in \Delta$ then $u = v$. Let u be such that $u - v \in \Delta$; then by Lemma 2.2 there exists a configuration w such that

$$w \overset{*}{\dashrightarrow} u \quad w \overset{*}{\dashrightarrow} v$$

Choosing a sufficiently large k we have

$$w + k\varepsilon \xrightarrow{*} u + k\varepsilon \quad w \xrightarrow{*} v + k\varepsilon$$

Then by Lemma 2.4, since u and v are recurrent : $u + k\varepsilon \xrightarrow{*} u$ and $v + k\varepsilon \xrightarrow{*} v$.

But u and v are two stable configurations obtained from $w + k\varepsilon$ by $\xrightarrow{*}$ they are equal by Proposition 2.1. □

An immediate corollary shows that the group $SP(G)$ may be considered from a different point of view :

Corollary 2.5. *For any graph G, the set of recurrent configurations equipped with the binary operation $\oplus$ is a group.*

In order to characterize the recurrent configurations D. Dhar introduced the configuration $\beta = -\Delta_n$; which is such that $\beta_j = e_{j,n}$ for $j \neq n$, and $\beta_n = -d_n$.

Corollary 2.6. Burning Algorithm. *The configuration u is recurrent if and only if*

$$u + \beta \xrightarrow{*} u$$

Moreover in this sequence of toppling each vertex topples exactly once.

Proof If $u + \beta \xrightarrow{*} u$ then u is recurrent since β is positive. Conversely let u be a recurrent configuration, then since $\beta = -\Delta_n$ we have :

$$u - (u \oplus \beta) \in \langle \Delta_1, \dots, \Delta_n \rangle <$$

Since u and $u \oplus \beta$ are recurrent we have $u = u \oplus \beta$ by Theorem 1.

Moreover

$$u + \beta = u + \sum_{i=1}^{n-1} \Delta_i$$

it is clear that each vertex x_i, $i = 1, \dots, n-1$ topples exactly once in $u + \beta \xrightarrow{*} u$. ☐

Example Let us return to the example of the 5-wheel and compute the set of the recurrent configurations. By the above characterization, a configuration $u = u_1, \dots, u_6$ is recurrent if and only if :

$$(u_1 + 1, u_1 + 1, u_1 + 1, u_1 + 1, u_1 + 1, u_6 - 5) \xrightarrow{*} u$$

This implies that at least one of the $u_i, i = 1, \dots, 5$ is equal to 2, moreover if $u_i = 0$ then the two neighbors of x_i in the cycle must topple before it, hence there are no two consecutive symbols 0 in the cycle $(u_1, u_2, u_3, u_4, u_5)$. For the same reason, between two occurrences of the symbol 0 in the cycle there cannot be a sequence of 1's. Hence the recurrent configurations are the following

- The 31 configurations with no $u_i = 0$: all the combinations of 1 and 2 are allowed except $(1, 1, 1, 1, 1)$.
- The 75 configurations containing exactly one 0 (5 possibilities for the position of 0, and 15 choices for all the other values since a sequence of 4 consecutive 1's is not allowed).
- The 15 configurations containing two 0's, which must have 2 and 2,1 or 2,2 between them.

These add up to 121 which is the order of the group $SP(W_5)$ computed above.

3 Sandpile groups of planar duals

If G be a planar connected multi-graph, and M a planar representation of G, a *dual* G^* of G is a multi-graph whose vertices are the faces of M (including the unbounded face), where to each edge e of G is associated an edge e^* of G^* connecting the two faces separated by e. A planar graph and its dual are represented in Figure 2 below.

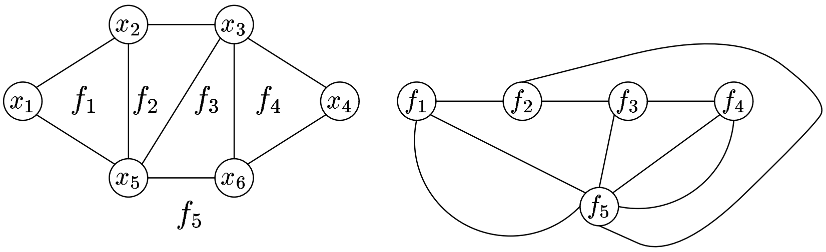

Figure 2: A planar graph and one of its duals

This section is devoted to the proof of the following :

Theorem 2. *For a planar graph* G *and any of its duals* G^* *the groups* $SP(G)$ *and* $SP(G^*)$ *are isomorphic.*

In order to prove this Theorem we use the *dart-space* of a multi-graph, also called edge-space([3]) or 1-chain group ([17]).

A dart of a multi-graph is an orientation of one of its edges; to each edge $\{x, y\}$ are associated two darts one from x to y and another from y to x, and these two darts are said to be *opposite.* Let $G = (X, E)$ be a multi-graph, and let D denote the set of darts; a *dart-configuration* is a mapping w of D to the set $\mathbb{Z}$ of integers such that for each pair of opposite darts a and b, the following relation is satisfied :

$$w(a) + w(b) = 0$$

We use the following convention in order to represent a dart-configuration, we draw an orientation of G and a labeling of the arcs; each edge is oriented in such a way that it represents the dart a such that $w(a) \geq 0$, the label of e is then $w(a)$; we use a line and not an arrow for edges whose darts a satisfy $w(a) = 0$.

A dart configuration of the graph above is given in Figure 3.

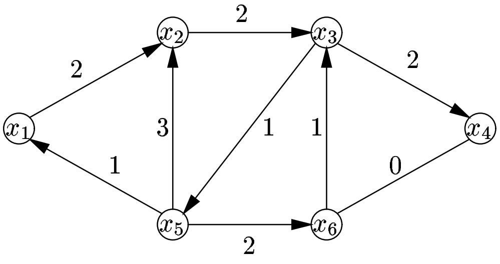

Figure 3: A dart configuration

To any dart configuration w of G is associated a (vertex) configuration ∂w of G and a (face) configuration $\partial^* w$ of G^* as follows :

- For any vertex x_i, $(\partial w)_i$ is equal to the algebraic sum of the $w(a)$ for all darts a with origin x_i.
- For any face f_j of G, $(\partial^* w)_j$ is the algebraic sum of $w(a)$ for all darts a bordering face f_j; in such a sum the darts are considered in the positive trigonometric orientation around face f_j, the map being embedded in the oriented sphere.

The configurations ∂w and $\partial^* w$ of w given in Figure 3 above are represented in Figure 4.

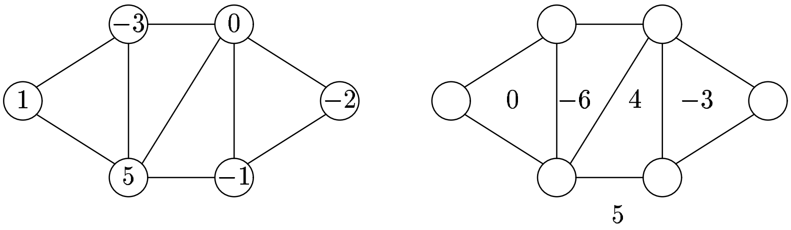

Figure 4: Configurations induced by a dart configuration

We first prove the following

Proposition 3.1. *For any configuration u on a connected multi-graph $G = (X, E, x_n)$, and such that*

$$\sum_{i=1}^{n} u_i = 0$$

there exists a dart configuration w satisfying :

$$\partial(w) = u$$

Proof Let u be such a configuration on the multi-graph G. Choose a spanning tree T of G with root x_n, and orient the edges of T in such a way that for any vertex x there is a directed path from x to the root x_n. Define w as follows

- For any dart a not in T, $w(a) = 0$.
- For a dart a whose origin x_i is a leaf of the tree $w(a) = u_i$
- For any vertex x_i such that $w(a)$ is defined for all the darts a entering it, define $w(b)$ for the unique dart b leaving x_i by :

  $$w(b) = u_i - \sum_{a} w(a)$$

  where the sum is taken over all the darts a with end point x_i.

Then the relation $\partial(w)_i = u_i$ is satisfied by construction for all vertices except possibly the root x_n, but since the sum of the u_i is 0 and

$$\sum_{a \in D} w(a) = 0$$

the relation is also satisfied for x_n $\square$

This construction also holds for any configuration v of G^* giving a w such that $\partial^* w$ is equal to v except for an arbitrary root vertex of G^* .

Let x_i be a vertex of G. We define the dart-configuration D_i such that $D_i(a) = 1$ if a starts from x_i; $D_i(a) = -1$ if a ends at x_i; and $D_i(a) = 0$ otherwise. It is easy to check that $\partial D_i = \Delta_i$ and $\partial^* D_i = 0$. The dart configuration D_j^* associated to any face f_j of a representation of a planar graph G is also of interest, it is defined by $D_j^*(a) = 1$ if a borders f_j in the positive orientation, $D_j^*(a) = -1$ if a borders it in the negative orientation; and $D_j^*(a) = 0$ otherwise. One can easily check that $\partial^* D_i = \Delta_i$ and $\partial D_i = 0$.

Two dart-configurations D_i and D_j^* for the graph above are represented in Figure 5.

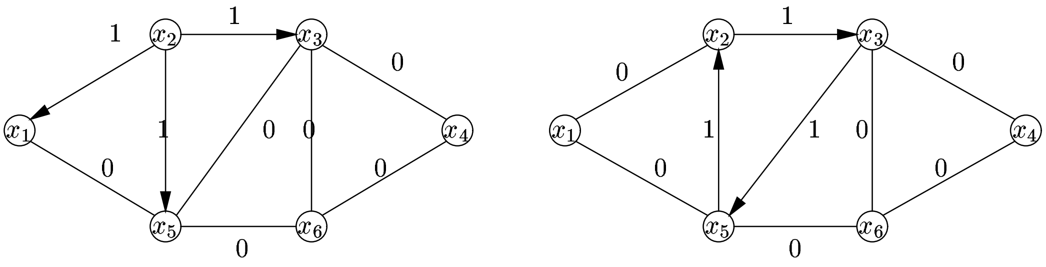

Figure 5: Configurations D_i and D_j^*

We will assume the following result which can be found (with a different notation) in [17] :

Proposition 3.2. *Let G be a planar graph and w a dart configuration of G such that $\partial w = 0$; then w is a linear combination of the D_j^*.*

Remark This result is often stated in the following and almost equivalent form "the faces of a planar map M form a basis for the cycle space of M", but here we assume more since the sums are over the ring $\mathbb{Z}$ instead of the usual situation were one considers vector spaces over a field (the complex numbers or $\mathbb{Z}/2\mathbb{Z}$).

Proof of the Theorem We define an isomorphism φ from $SP(G)$ to $SP(G^*)$ as follows :

Let $\overline{u}$ be an element of $SP(G)$ and u a configuration of the multi-graph G such that $\overline{u}$ is the image of u and $\sum_{i=1}^n u_i = 0$. Then there exists a dart-configuration w such that $\partial w = u$, and we define $\varphi(\overline{u})$ by :

$$\varphi(\overline{u}) = \overline{\partial^* w}$$

where $\overline{\partial^* w}$ is the image of $\partial^* w$ in $SP(G^*)$.

We first have to prove that φ is well-defined since we made two arbitrary choices in the above definition : the first choice was to take a u whose image is $\overline{u}$ in $SP(G)$, the second was the choice of w such that $u = \partial w$. For this second choice note that if w' is such that $\partial w = \partial w' = u$ then by the above Proposition $w - w'$ is an element of $\langle D_1^*, \cdots, D_p^* \rangle$ hence

$$\partial w - \partial w' \in \langle \Delta_1^*, \Delta_2^*, \ldots, \Delta_p^* \rangle$$

and $\overline{\partial w} = \overline{\partial w'}$. Consider the first choice, and let u and u' be two configurations on G such that $\overline{u} = \overline{u}'$, then $u = u' + \sum \alpha_i \Delta_i$. Consider w such that $\partial w = u$; it can easily be checked that $\partial(\underline{w + \alpha_i D_i}) = u'$, but $\partial^* D_i = 0$ hence $w' = w + \alpha_i D_i$ defines $\varphi(u')$ and thus $\overline{\partial^* w} = \overline{\partial^* w'}$. This proves the correctness of the definition of φ.

Let us prove that φ is a homomorphism of groups : for two configurations u and v of G and two dart-configurations such that $\partial w = u$, $\partial t = v$ we have $\partial(w + t) = u + v$, and hence :

$$\varphi(\overline{u} + \overline{v}) = \partial^*(w + t) = \partial^* w + \partial^* t$$

The homomorphism φ is one-to-one since by a symmetric construction we can associate to any configuration v^* on G^* a dart configuration w^* such that $\partial^* w^* = v^*$ and $\varphi(\overline{\partial w^*}) = v^*$. ☐

4 Group of the n-wheel

The n-wheel W_n is a graph with $n + 1$ vertices, n of which form a cycle and the last one is joined by an edge to each of the others. The Δ_i for the n-wheel are given for $i = 2, \cdots, n - 1$ by :

$$\Delta_i = 3x_i - x_{i+1} - x_{i-1} - x_{n+1}$$

and

$$\Delta_1 = 3x_1 - x_2 - x_n - x_{n+1}, \quad \Delta_n = 3x_n - x_1 - x_{n-1} - x_{n+1}$$

In the group $\mathbb{Z}^{n+1}/\Delta(G, x_{n+1})$ we have the relations for $n = 3, \ldots, n$:

$$\overline{x}_i = 3\overline{x}_{i-1} - \overline{x}_{i-2}$$

Thus we can express $\overline{x}_i$ in terms of $\overline{x}_1$ and $\overline{x}_2$ by a formula of the form :

$$\overline{x}_i = \alpha_i \overline{x}_2 + \alpha_{i-1} \overline{x}_1$$

This gives the following recursion formula for the α_i :

$$\alpha_2 = 1, \quad \alpha_3 = 3, \quad \alpha_i = 3\alpha_{i-1} - \alpha_{i-2} \text{ for } \ i > 3$$

and we also have the two relations :

$$\overline{x}_1 = \alpha_{n+1} \overline{x}_2 + \alpha_n \overline{x}_1 \qquad \overline{x}_2 = \alpha_{n+2} \overline{x}_2 + \alpha_n \overline{x}_1$$

It is easy to check that the α_i satisfy the same recurrence relation as the even terms of the Fibonacci numbers f_n, since from $f_{i+2} = f_{i+1} + f_i$ we get :

$$f_{i+2} = 2f_i + f_{i-1} = 3f_i - f_{i-2}$$

Since $f_2 = \alpha_2 = 1$ and $f_4 = \alpha_3 = 3$, we have

$$\alpha_n = f_{2n-2}$$

The relations between $\overline{x}_1$ and $\overline{x}_2$ become :

$$\left\{ \begin{array}{ccccc} f_{2n}\overline{x}_2 & - & (f_{2n-2}+1)\overline{x}_1 & = 0 \\ (f_{2n+2}-1)\overline{x}_2 & - & f_{2n}\overline{x}_1 & = 0 \end{array} \right.$$

From this we obtain the following :

Theorem 3. *The sandpile group of the n-wheel W_n is the direct product of two cyclic groups. If n is even, these two groups are of order $5f_n$ and f_n, and if n is odd they are both of order $f_{n-1} + f_{n+1}$.*

Proof We have to determine the Smith normal form of the matrix :

$$M_n = \begin{pmatrix} f_{2n} & -(f_{2n-2}+1) \\ & \\ f_{2n+2}-1 & -f_{2n} \end{pmatrix}$$

We use the two classical identities (see for instance [11] for details) :

$$f_{2n} = f_n(f_{n-1} + f_{n+1})$$

and (Cassini's identity) :

$$f_n f_{n-3} = f_{n-1} f_{n-2} + (-1)^n$$

These give :

$$f_{2n+2} - 1 = \begin{cases} f_n f_{n+1} + f_n f_{n+3} & \text{if } n \text{ is even} \\ f_{n+1} f_{n+2} + f_{n-1} f_{n+2} & \text{if } n \text{ is odd} \end{cases}$$

and

$$f_{2n-2} + 1 = \begin{cases} f_n f_{n-1} + f_n f_{n-3} & \text{if } n \text{ is even} \\ f_{n+1} f_{n-2} + f_{n-1} f_{n-2} & \text{if } n \text{ is odd} \end{cases}$$

When n is odd we get :

$$M_n = \begin{pmatrix} f_{n-1} + f_{n+1} & 0 \\ 0 & f_{n-1} + f_{n+1} \end{pmatrix} \begin{pmatrix} f_n & f_{n-2} \\ f_{n+2} & f_n \end{pmatrix}$$

and the result follows from the computation of the determinant

$$\begin{vmatrix} f_n & f_{n-2} \\ f_{n+2} & f_n \end{vmatrix} = \begin{vmatrix} f_n & f_{n-2} \\ f_{n+1} & f_{n-1} \end{vmatrix}$$

which is equal to 1 by Cassini's identity.
When n is even, we have :

$$M_n = \begin{pmatrix} f_n(f_{n+1} + f_{n-1}) & f_n(f_{n-1} + f_{n-3}) \\ f_n(f_{n+1} + f_{n+3}) & f_n(f_{n+1} + f_{n-1}) \end{pmatrix}$$

Adding the second line to the first we obtain:

$$M'_n = \begin{pmatrix} f_n(2f_{n+1} + f_{n-1} + f_{n+3}) & f_n(2f_{n-1} + f_{n-3} + f_{n+1}) \\ f_n(f_{n+1} + f_{n+3}) & f_n(f_{n+1} + f_{n-1}) \end{pmatrix}$$

and :

$$M'_n = \begin{pmatrix} 5f_n & 0 \\ 0 & f_n \end{pmatrix} \begin{pmatrix} f_{n+1} & f_{n-1} \\ f_{n+1} + f_{n+3} & f_{n+1} + f_{n-1} \end{pmatrix}$$

But the determinant of the second matrix is equal to

$$\begin{vmatrix} f_{n+1} & f_{n-1} \\ f_{n+3} & f_{n+1} \end{vmatrix} = \begin{vmatrix} f_n & f_{n-1} \\ f_{n+2} & f_{n-1} \end{vmatrix}$$

which is equal to 1 by Cassini's identity.

5 Groups of complete graphs

In the complete graph K_n, the configurations Δ_i, $1 \leq i \leq n-1$ are given by :

$$(n-1)x_i - (x_1 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_n)$$

Replacing Δ_{n-1} by the sum of all these and adding sufficiently many times x_n we get

$$x_1 + x_2 + \cdots + x_n$$

But this added to each Δ_i gives

$$\forall i, \; i = 1, \ldots n-2 \qquad nx_i \in \Delta(K_n, x_n)$$

We thus get in concordance with Cayley's formula :

Theorem 4. *The sandpile group of the complete graph K_n is the direct product of $n-2$ cyclic groups of order n.*

We show now how the recurrent configurations of K_n are related to parking functions. Recall the definition of the parking functions

Definition 2. *A sequence $t_1, t_2, \ldots, t_n$ of non-negative integers is an n-parking function if there exists a permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ such that*

$$\forall i = 1, \ldots, n, \quad a_i \geq t_i$$

The relationship with recurrent configurations in the complete graph is given by :

Proposition 5.1. *The configuration $u_1, \ldots u_n, u_{n+1}$ of the complete graph K_{n+1} is recurrent if and only if the sequence*

$$n - u_1, n - u_2, \ldots, n - u_n$$

is an n-parking function.

Proof By Corollary 2.6, the configuration $u_1, u_2, \ldots u_n, u_{n+1}$ is recurrent if and only if

$$(u_1 + 1, u_2 + 1, \ldots u_n + 1, u_{n+1} - n) \xrightarrow{*} u$$

and in this sequence of toppling each vertex topples exactly once. So we can define for each instant t $1 \leq t \leq n$ a vertex toppling at instant t, for $i = 1, \ldots n$, let a_i be the instant in which vertex i topples, the a_i's are not determined uniquely but any such sequence is valid. When x_i topples it had $u_i + 1$ grains at the beginning of the toppling process, and has received $a_i - 1$ from the vertices toppling before it, thus we have :

$$u_i + 1 + (a_i - 1) \geq n$$

Simplifying this gives $a_i \geq n - u_i$ so that $n - u_i$ is an n-parking function. The converse is immediate. □

Let us consider now the *complete bipartite graph $K_{n,n}$* where the vertices are denoted by $x_1, \ldots, x_n, y_1, \ldots, y_n$ and the root is y_n. There are $2n$ elements Δ_i which we write

$$\begin{aligned} \Delta'_i &= ny_i - (x_1 + \cdots + x_n) \\ \Delta''_i &= nx_i - (y_1 + \cdots + y_n) \end{aligned}$$

Computing the Smith Normal Form of the matrix determined by these relations gives :

Theorem 5. *The sandpile group of the complete bipartite graph $K_{n,n}$ is the direct product of $2(n-2)$ cyclic groups of order n and a cyclic group of order n^2.*

Proof We consider the Δ'_i for $i = 1, \ldots, n$ and the Δ''_i for $i = 1, \ldots, n-1$. In the quotient group $SP(K_{n,n})$ replacing Δ''_{n-1} with the sum of all relations we get

$$\overline{x}_1 + \overline{x}_2 + \cdots + \overline{x}_n = 0$$

Hence adding this relation to all the Δ'_i and subtracting Δ''_1 from all the Δ''_i, $i = 2, \ldots, n-2$ we obtain

$$\begin{cases} n\overline{y}_i = 0 & i = 1, \ldots, n-1 \\ \overline{x}_1 + \overline{x}_2 + \cdots + \overline{x}_n = 0 & \\ \overline{y}_1 + \overline{y}_2 + \cdots + \overline{y}_{n-1} - n\overline{x}_1 = 0 & \\ n(\overline{x}_i - \overline{x}_1) = 0 & i = 2, \ldots, n-2 \end{cases}$$

Consider the relation $n\overline{y}_{n-1} = 0$; this may be replaced with

$$n(n\overline{x}_1 - \overline{y}_1 - \overline{y}_2 - \cdots \overline{y}_{n-2}) = 0$$

But $n\overline{y}_i = 0$ for any $i = 1, \dots, n-1$ giving

$$n^2 \overline{x}_1 = 0$$

We end the proof by using as generators the following $2n-1$ elements

$$x_1, x_2 - x_1, \dots, x_{n-1} - x_1, y_1, \dots, y_{n-2}, x_1 + \cdots x_n, y_1 + \cdots + y_{n-1} - nx_1$$

It is easy to check that they are indeed generators and their orders in $SP(K_{n,n})$ are : n^2 for the first one, n for $2n-4$ of them and 1 for the last two. □

ACKNOWLEDGMENTS

The authors wish to thank O. Marguin ([14]) who introduced the subject to them, and also A. Machì and G. Schaeffer for many fruitful discussions.

References

[1] P. Bak. *How Nature works : the science of self-organized criticality.* Springer-Verlag, 1996.

[2] P. Bak, C. Tang, and K. Wiesenfeld. Self-organized criticality. *Phys. Rev. A (3)*, 38(1):364–374, 1988.

[3] N. Biggs. *Algebraic graph theory.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 1993.

[4] A. Björner, L. Lovász, and P. Shor. Chip-firing games on graphs. *European J. Combin.*, 12(4):283–291, 1991.

[5] H. Cohen. *A course in computational algebraic number theory*, volume 138 of *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, 1993.

[6] D. Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.*, 64(14):1613–1616, 1990.

[7] D. Dhar, P. Ruelle, S. Sen, and D. Verma. Algebraic aspects of abelian sandpile models. *Journal of Physics A***28**, 28:805–831, 1995.

[8] D. Foata and J. Riordan. Mappings of acyclic and parking functions. *Aequationes Math.*, 10:10–22, 1974.

[9] A. Gabrielov. Abelian avalanches and Tutte polynomials. *Phys. A*, 195(1-2):253–274, 1993.

[10] E. Goles. Sand piles, combinatorial games and cellular automata. In *Instabilities and nonequilibrium structures, III (Valparaíso, 1989)*, volume 64 of *Math. Appl.*, pages 101–121. Kluwer Acad. Publ., Dordrecht, 1991.

[11] R. Graham, D. Knuth, and O. Patashnik. *Concrete mathematics.* Addison-Wesley Publishing Company, Reading, MA, second edition, 1994. A foundation for computer science.

[12] J. Hafner and K. McCurley. Asymptotically fast triangularization of matrices over rings. *SIAM J. Comput.*, 20(6):1068–1083, 1991.

[13] N. Jacobson. *Basic algebra. II.* W. H. Freeman and Company, New York, second edition, 1989.

[14] O. Marguin. *Application de méthodes algébriques à l'étude algorithmique d'automates cellulaires et à l'analyse par ondelettes.* PhD thesis, Université Claude Bernard, Lyon 1, 1997.

[15] R. Stanley. Parking functions and noncrossing partitions. *Electron. J. Combin.*, 4(2):Research Paper 20, approx. 14 pp. (electronic), 1997. The Wilf Festschrift (Philadelphia, PA, 1996).

[16] R. Stanley. Hyperplane arrangements, parking functions and tree inversions. In *Mathematical essays in honor of Gian-Carlo Rota (Cambridge, MA, 1996)*, volume 161 of *Progr. Math.*, pages 359–375. Birkhäuser Boston, Boston, MA, 1998.

[17] W. T. Tutte. *Graph theory*, volume 21 of *Encyclopedia of Mathematics and its Applications.* Addison-Wesley Publishing Co., Reading, Mass., 1984. With a foreword by C. St. J. A. Nash-Williams.